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Evolution of Darwinian drift

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The evolution of drift with time at a plane normal to the passage of a body from far right to far left is investigated. It is shown that the measurable part of the drift volume as a function of time is uniquely determined, and that the indeterminacy of Darwin's drift integral simply does not matter to the physical world, in which there is a Darwin theorem stating that the drift mass is equal to the added mass. The method for determining the shape of the drift surface is also given.

1. Introduction

Darwin (1953) studied fluid drift in any unbounded irrotational flow caused by a solid body in steady translation. In the same year Ursell (1953) and Longuet-Higgins (1953) investigated mass transport in water waves. Darwin's paper, now famous, contains the fascinating result that the drift mass is equal to the added mass. Perhaps stimulated by that result, Lighthill (1956) soon afterwards published a paper on drift in shear flows, in the first issue of this journal.

On the side that concerns added mass, Taylor (1928) published a result on the relation between the added mass for the irrotational flow caused by a translating solid body and the singularities inside the body that create the flow. Taylor's result was generalized by Birkoff (1950) and Landweber (1956) into a theorem that now bears his name. In retrospect, it is remarkable that the few years 1950–56 saw so many interesting papers on drift and added mass published.

Darwin's famous result has been, from the beginning, plagued by the fact that his drift integral, which is prima facie a momentum integral (for unit speed of the body) does not have a unique value. Darwin (1953) found that if that integral is evaluated longitudinally first, the drift mass m_d is equal to the added mass m_a. This result I call the Darwin theorem. But other ways of evaluating the integral give other results. Integrating transversely (to the passage of the body) first gives the drift mass $-\rho V$, for instance, where ρ is the density of the fluid and V the volume of the body.

Darwin recognized that the difference between the two results for m_a and $-\rho V$ is spread over a very wide area ('At edges', he said), and thus reconciled the two procedures of obtaining the drift mass. Benjamin (1986), in a paper criticizing mine (Yih 1985) for not mentioning the indeterminacy of Darwin's drift integral, which he emphasized, also corrected Darwin in a footnote about 'the edges', and said the difference is spread 'uniformly' over the cross-section. Actually it is not spread uniformly either, but accordingly as the vanishing magnitude of the velocity potential (denoted by ϕ' herein), which is a function of the transverse coordinates. We shall show unambiguously how this difference comes to be *infinitely* widely spread.

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More importantly, however, we shall treat the drift as a function of time, as the body moves from $x' = \infty$ to $x' = -\infty$ with unit speed, and seek to determine the volume and shape of that drift. This has not been attempted before.

The question of indeterminacy of the drift integral was investigated by Eames, Belcher & Hunt (1994). They first studied the partial drift at a plane due to a sphere that started to move at a finite distance x_0 to the right of that plane, and later generalized their result to apply to a body of arbitrary shape. They reached the same conclusion that any difference between the drift volume and m_a/ρ (for large x_0) is widely spread, and their analysis and calculations contributed substantially in illustrating the indeterminancy of the drift integral and in explaining the intricacy of the Darwin problem.

With (x', y', z') denoting Cartesian coordinates in the fixed frame of reference, we seek to determine the drift in the (y', z')-plane as a function of time t. Clearly the drift there at the time when the body reaches the position $x' = x_0$ is precisely what Eames *et al.* removed in determining their partial drift, and the removed part is not small if x_0 is not large. In any event our aim and point of view are different from those of Eames *et al.* Specifically, we shall

(i) show what is the measurable part of the drift as a function of time;

(ii) show how the distribution of the drift can be determined, thereby revealing the *infinitely* widely distributed difference of the drift volume (obtained by transverse integration first) from m_a/ρ, noted by Darwin, Benjamin, and Eames *et al.*; and

(iii) conclude, as a consequence, that to the physical world, or any experimenter or any positivist follower of Auguste Comte, who would exclude the 'unknowable' from consideration, there is a Darwin theorem.

Instead of determining the drift in the (y', z')-plane per se, we take the useful and expedient alternative of considering the steady flow relative to the body and determining the volume and shape of the fluid surface, dyed red (say), initially coinciding with a plane at $x' = -\infty$ and normal to the x'-axis, as it sweeps toward the right. By volume we mean the volume† of the *fluid* between the red surface and the plane tangent to it at infinity. This volume is exactly the drift volume at the (y', z')-plane, which we sought originally. The chief distinction between streamwise integration first and transverse integration next is in the values of the integral of ϕ' (velocity potential in the fixed frame) at infinite distance from the body. We shall show that in two extreme procedures the measurable part of the drift is always the same at *any time*, making the difference in the drift-volume values obtained by the two procedures immaterial to the physical world.

In §2, some preliminaries will be given to facilitate later developments, and in §3 we shall show where the net momentum resides in the two extreme procedures of integration. This will be useful to us for our main arguments in §4, where we shall reach our main conclusion.

† It is important to keep in mind that this volume is that of the fluid bounded by the red surface, the plane to which it is tangent at infinity, and the surface of the body if the body is passing or has passed through the (y', z')-plane. This remark is consistent with all the developments in this paper. The red surface will, as time passes, wrap more and more tightly around the body, eventually closing at the rear stagnation point, which will eventually be infinitely far to the left of the (y', z')-plane. Nevertheless the drift volume defined herein is integrable and finite at all times. These facts show the intricacy of the Darwin problem. They also show the advantage of using the steady flow in the moving frame, thus avoiding the particle trajectories in the fixed frame, with their fascinating but somewhat distracting loops, and making the drift volume more visualizable.

2. Preliminaries

If a body moves with velocity U in translation, the kinetic energy (KE) and the momentum (M) of the fluid can be written as

$$\mathrm{KE} = \tfrac{1}{2} m_k U^2, \quad M = m_m U,$$

where m_k is the kinetic-energy mass and m_m the momentum mass. It is well-known that m_k is the added mass, commonly denoted by m_a. The reason for the name 'added mass' is as follows.

If U varies with time t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{KE} = m_k U \frac{\mathrm{d}U}{\mathrm{d}t} = F_1 U,$$

where F_1 is the component of the force in the direction of U, with which the body is doing work to increase the kinetic energy of the fluid. Thus

$$F_1 = m_k \frac{\mathrm{d}U}{\mathrm{d}t}.$$

If the body is symmetric with respect to an axis in the direction of motion, F_1 is the only non-zero component of the force, and to a person trying to accelerate a body in a fluid, the fluid seems to provide an added mass to the body. That added mass is exactly m_k, as shown by the equation above. It has been denoted by m_a in the literature, and will be so denoted here. As to the momentum mass m_m, it is always equal to the drift mass m_d, which we shall define later. But the momentum M is indeterminate (Theodorsen 1941; Birkhoff 1950; Darwin 1953), and that is the burden of this paper.

We shall let $U = -1$, so that relative to the body the velocity of the fluid at infinity is equal to 1. The momentum M will be negative, but m_m and m_d will be positive. We can, to fix ideas, regard the drift mass for any U to be

$$\rho|\text{drift volume}|\, U,$$

which can be positive or negative. Our m_d is then simply

$$m_d = \rho|\text{drift volume}|. \tag{1}$$

We shall not use m_m explicitly, and shall deal with M directly.

We shall treat the three-dimensional case only, since reduction to the two-dimensional case can be easily done. Using unit speed amounts to using the body speed as velocity scale. In a frame moving with the body, we use the coordinates (x, y, z), which are related to the coordinates in the fixed frame by

$$x = x' + t, \quad y = y', \quad z = z'. \tag{2}$$

The velocity components in the two frames will be denoted by

$$(u', v', w') \quad \text{and} \quad (u, v, w),$$

respectively. These are related by

$$u = u' + 1, \quad v = v', \quad w = w'. \tag{3}$$

The speeds q' and q are defined by

$$q'^2 = u'^2 + v'^2 + w'^2, \quad q^2 = u^2 + v^2 + w^2. \tag{4}$$

Since the flow is assumed irrotational, we have a velocity potential, which will be denoted by ϕ' and ϕ, respectively, in the two frames, related by

$$\phi = x+\phi'. \tag{5}$$

Darwin's definition of the drift volume (recall that u' is for unit body speed) is

$$M/\rho = \iiint u'\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z', \tag{6}$$

which is prima facie a momentum integral divided by the density ρ. The connection between momentum and drift is through the discharge at $x' = 0$ when the body moves from $x' = \infty$ to $x = -\infty$. To see the connection, note first that ϕ' and the velocity components are functions only of (x, y, z). At $x' = 0$, we obtain from (1)

$$\mathrm{d}x = \mathrm{d}t,$$

and immediately the integral (6) becomes

$$\iint\left[\int_{-\infty}^{\infty} u'\,dt\right]\mathrm{d}y\,\mathrm{d}z, \tag{7}$$

which is clearly the discharge through $x' = 0$ as t varies from $-\infty$ to ∞. But this discharge is exactly the drift volume, and that is why Darwin defined the drift volume by that integral. With this understanding the equality of the drift mass, which is kinematic in nature, and the added mass, which is a kinetic-energy mass, is no longer as puzzling as at first sight, for the connection is through the momentum, which is a dynamical quantity.

But the drift volume given by (7) is a total discharge. It does not reveal the drift distance of each fluid particle. A finer understanding of (6) can be had by writing (Yih 1995)

$$I = I_1 - I_2, \tag{8}$$

where

$$I = \iiint[(u-1)^2+v^2+w^2]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint(q'/q)^2\,\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\chi, \tag{9}$$

$$I_1 = \iiint(1-u)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\left(\frac{1}{q^2}-\frac{\partial\chi}{\partial\phi}\right)\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\chi, \tag{10}$$

$$I_2 = \iiint(u-q^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\left(\frac{\partial\chi}{\partial\phi}-1\right)\mathrm{d}\phi\,\mathrm{d}\psi\,\mathrm{d}\chi. \tag{11}$$

These differ from the integrals denoted by the same symbols in Yih (1995) only by a factor ρ. Clearly I_1 is the negative of (6). We use I_1 because for the body moving left it is positive. The ψ and χ are stream functions for the flow in the moving frame of reference. These are related to the stream functions (ψ', χ') in the fixed frame by

$$\psi = \tfrac{1}{2}r^2+\psi', \quad \chi = \chi', \quad r^2 = y^2+z^2. \tag{12}$$

In (10) and (11)

$$\frac{\partial\chi}{\partial\phi} = \frac{u}{q^2}, \quad q^2 = \frac{\partial(\phi,\psi,\chi)}{\partial(x,y,z)}, \tag{13}$$

since

$$(u,v,w) = (\phi_x,\phi_y,\phi_z) = \operatorname{grad}\psi\times\operatorname{grad}\chi. \tag{14}$$

Let us look at the inner integral of I_1, which is

$$I_3 = \int \left(\frac{1}{q^2} - \frac{\partial \chi}{\partial \phi} \right) \mathrm{d}\phi. \tag{15}$$

This is the drift distance if the limits of integration are $-\infty$ and ∞. The first term on the right-hand side is the time required by a fluid particle to go from $\phi = -\rho$ to $\phi = \infty$, for any fixed values of ψ and χ, whereas the second term without the minus sign is just the time required by a kinematic point of velocity 1 to do the same. At infinity $u = 1$. Therefore, the time difference is the drift distance. We shall, in §4, relate the time difference to the drift distance when considering drift at a finite time instead of an infinite t.

3. Integrating the drift integral

The value of the drift integral depends on the shape of the fluid boundary at infinity, as mentioned before. For the complete Darwin drift and in the moving frame of reference, we have for the geometry of the fluid two choices:

Choice A: a rectangle of sides $2X$ and $2Y$, enclosing the body in the middle, with X, Y, and X/Y approaching infinity; and

Choice B: the same, but with X, Y, and Y/X approaching infinity.

There are, of course, many rectangular shapes between Choices A and B. But treating the two extreme cases will suffice to make our point.

In practice, the choices defined above can take other forms. For instance, integrating in an infinite domain with respect to x first, or streamwise first, amounts to taking Choice A. Doing the opposite amounts to taking Choice B. Taking a domain infinite longitudinally but limited transversely, integrating in any order, and then letting the transverse extent expand to infinity amounts to adopting Choice A, whereas doing the opposite is adopting Choice B. In §4 we shall show that the region of integration is specified for us, but that there are still two ways of integrating, which amount to A and B. But first we shall obtain some results by adopting A and B in turn. These are interesting and will shed some light on the effect of the procedure of summing momentum on the result obtained. They will also be useful in §4, when we shall determine the drift volume and shape as functions of time.

3.1. *Choice A*

Under Choice A, two results are obtainable without ambiguity or controversy. Consider the domain in a stream tube extending from $x = -\infty$ to $x = \infty$. Then

$$I_2 = \iiint \left(\frac{\partial \chi}{\partial \phi} - 1 \right) \mathrm{d}\phi \, \mathrm{d}\psi \, \mathrm{d}\chi = \iint [\phi'(\infty) - \phi'(-\infty)] \, \mathrm{d}\psi \, \mathrm{d}\chi = 0.$$

The order of integration is now not in dispute, since the tube has finite cross-section. Thus (8) gives, after recalling that the body is supposed to have unit speed,

THEOREM 1. *Twice the kinetic energy in any stream tube of finite cross-section extending from $x = -\infty$ to $x = \infty$ in the steady flow relative to the body is equal to the momentum in that tube times the body velocity, and to the drift mass times the square of the body velocity.*

If we integrate over the entire fluid domain, we have Darwin's theorem, but we also inherit his dilemma. Now let the stream tube shrink to a streamline. Then (8) gives

$$\int (q'/q)^2 \,\mathrm{d}\phi = \int \left(\frac{1}{q^2} - \frac{\partial \chi}{\partial \phi}\right) \mathrm{d}\phi,$$

with limits of integration $(-\infty, \infty)$ for both integrals. The right-hand side is the negative of the drift distance which in our case is to the left. Thus we have

THEOREM 2. *For the body moving to the left* (*right*), *the drift distance is always negative* (*positive*).

Note that (15) gives the time delay, which is the negative of the drift distance only because at $x = \infty$ the component u is 1. Theorems 1 and 2 support the priority of Choice A for getting the complete drift volume of Darwin.

We shall now show where the net momentum resides if we take Choice A. Let the surface of the body be denoted by S_B, and the boundary of the projection of S_B onto the (y, z)-plane be denoted by C. The cylindrical surface with generatrix parallel to the x-axis and intersecting the (y, z)-plane at C, extending from $x = -\infty$ to $x = \infty$, will be denoted by S_C. The space bounded by S_B and S_C will be denoted by D, and the cylinder will be called the contact cylinder.

Integrating (10) in D, we obtain

$$I_1 = \iint \phi'_B n_1 \,\mathrm{d}S_B, \tag{16}$$

where (n_1, n_2, n_3) are the components of the unit vector drawn into the fluid on S_B. This is exactly the m_a/ρ for translation with unit speed in the direction of decreasing x. See, for instance, Landweber (1956) or Landweber & Yih (1956, equation (24)). Therefore

$$\rho I_1 = m_a. \tag{17}$$

Now construct any cylinder parallel to the x-axis and containing the contact cylinder. Obviously the integration in the x-direction for the momentum between the two cylinders gives zero as the result, since ϕ' vanishes at infinity. Thus we have the following results.

The momentum within any cylinder parallel to the x-axis and containing the contact cylinder and, in particular, within the contact cylinder itself, is exactly $-m_a$, since the body moves with unit speed to the left.

This result must have been known to Benjamin (1986), although he cast it in terms of discharge through a hole in the (y, z)-plane. As a result for mass discharge, it is entirely correct. But then he interpreted this discharge as the drift mass. That can only mean that, if one dyes the fluid red in the circular area $r = R$ (R is Benhamin's notation for the radius of the 'hole') at $x' = 0$, and lets the body move from $x' = -\infty$ to $x' = \infty$ according to Benjamin's scheme, the volume under the red cap, or the volume bounded by it, the (y', z')-plane, and the cylindrical wall projecting the red cap onto that plane, will, when the body reaches $x' = \infty$, be the drift volume. But there is flow through the wall, so that the said volume will not be the total discharge from $t = -\infty$ to $t = \infty$, and will therefore not be m_a/ρ. The error disappears only if $R = \infty$, for which one obtains $m_d = m_a$ by integrating with respect to x first. But then Benjamin would be using the very procedure he was criticizing. He was too consistent for that. Thus the error remains. The footnote on p. 253 of Benjamin's (1986) paper indicates strongly that he was aware of this difficulty, but being aware of it was not to resolve it. Eames *et al.* also claim that Benjamin was in error. But I do not agree with their statement that the error is traceable to the passage from Benjamin's equation (5) to his (6). After all, a tube of finite radius was being considered by Benjamin. The passage from his (3), (4), and (5) to (6) is correct, and so is his (6).

Equation (17), interesting as it is, does not give the distribution of the drift distance. To find that we have to use (15).

3.2. *Choice B*

In the moving frame, let two planes

$$x = x_1 \quad \text{and} \quad x = x_2$$

just enclose the body, so that x_1 is negative and x_2 positive. The momentum of the fluid in the domain bounded by these planes and the body can be evaluated in two ways. The results are the same, since the domain has finite width. Integrating with respect to x first, we have, by virtue of (16) and (17),

$$I_1 = -I_R + I_L + \iint \phi' n_1 \, \mathrm{d}s_B = -I_R + I_L + m_a/\rho, \tag{18}$$

where $$I_R = \iint \phi'(x_2) \, \mathrm{d}y \, \mathrm{d}z \quad \text{and} \quad I_L = \iint \phi'(x_1) \, \mathrm{d}y \, \mathrm{d}z,$$

and the dependence of ϕ' on y and z is implied.

On the other hand, if we integrate with respect to y and z first, we have

$$I_1 = -\iiint \frac{\partial(\psi', \chi')}{\partial(y, z)} \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}x = \int Q \, \mathrm{d}x \tag{19}$$

where $$Q = -\iint \mathrm{d}\psi' \, \mathrm{d}\chi'.$$

At any x, as we integrate from the body outward to infinity, where $\psi' = 0$, the inner integral Q is

$$Q = \int_0^{2\pi} \psi'(x) \, \mathrm{d}\chi'(x).$$

But this is exactly

$$\iint \mathrm{d}\psi'_B \, \mathrm{d}\chi'_B$$

over the part of S_B from x_1 to x, and is therefore the discharge across that part of S_B, which is, by the boundary condition of the flow on S_B, exactly equal to $-A(x)$, where $A(x)$ is the cross-sectional area of the body at x. Hence

$$I_1 = -\int A(x) \, \mathrm{d}x = -V. \tag{20}$$

Recalling that ρI_1 is the negative of the momentum, this means that the momentum is V for body velocity -1, which is also the drift mass, now a reflux.

Equating (19) and (20), we see that

$$I_R - I_L = V + m_a/\rho. \tag{21}$$

Now take any plane

$$x = x_3 < x_1.$$

It can be shown as in the foregoing that the momentum between the planes at x_3 and x_1 is zero. Therefore

$$I_L(x_1) = I_L(x_2). \tag{22}$$

Similarly $$I_R(x_2) - I_R(x_4) \quad \text{for} \quad x_4 > x_2. \tag{23}$$

Equations (22) and (23) are true however large $-x_3$ and x_4 are. The conclusion for Choice B, then, is that the net momentum resides in the space bounded by the planes touching the body. But even in the limit, at infinity, the integrals I_L and I_R are not zero.

The results according to Choice B may be counter-intuitive. For instance, equation (20), however understandable from the point of view of drift, seems strange from the dynamical point of view, since the integral also represents momentum. Another result that seems strange is that a plate of no thickness moving broadside-on will give the fluid no momentum, according to Choice B. But, strange as they may seem, no law of mechanics is violated, even if one considers the motion to have started from rest. (In that case there is a net force at infinity for Choice B.) Thus there does not seem to be any hope at all that a rigorous proof exists for a unique way of evaluating the momentum integral.

We have seen that the momentum between any two planes both to the left or both to the right of the body is zero. The momentum in

$$-\infty \leqslant x \leqslant \xi, \quad x_1 \leqslant \xi \leqslant x_2$$

and extending to infinity laterally is equal to $V(\xi)$, which denotes the volume of the body to the left of $x = \xi$. For $\xi \geqslant x_2$, the total momentum up to $x = \xi$ is always ρV. These results will be useful for the next section.

4. Calculation of drift as a function of time

We recall that for the entire fluid we can go from the momentum integral (6) to the integral (7), thereby interpreting (6) as the drift integral as well. Suppose now the range of integration is

$$-\infty \leqslant x \leqslant \xi. \tag{24}$$

Then, since $x = x' + t$, at $x' = 0$ the right-hand side of (6) converts to

$$\iiint \left[\int_{-\rho}^{\xi} u' \, \mathrm{d}t \right] \mathrm{d}y \, \mathrm{d}z, \tag{25}$$

which is the drift from the (y, z')-plane in the period $-\infty \leqslant t \leqslant \xi$, or the drift when the (centroid of the) body is at distance $-\xi$ to the right of the (y', z')-plane. It is also the drift volume under the red surface defined in this section, since the body is at the distance $-\xi$ to the right of the plane where drift is calculated. The study of the evolution of drift is then reduced to the evaluation of the momentum integral I_1 ($-\rho I_1 =$ momentum), in the steady flow relative to the body.

The region of interest for time evolution of drift is then (24). We shall let the transverse extent (for ϕ' and $x = -\infty$) be infinite to start with. We can still distinguish

Choice A from Choice B, because if one integrates streamwise first, one has $\phi' = 0$ at $x = -\infty$, and its integral with respect to y and z will be zero, whereas if one adopts Choice B one has

$$\lim_{X\to\infty} \int \phi'(X, y, z)\,\mathrm{d}y\,\mathrm{d}z = -\tfrac{1}{2}(V + m_a/\rho), \tag{26}$$

which is related to the Taylor theorem mentioned before. But that is the *only* difference. Streamwise integration is necessary for both choices, since the calculation of drift is inherently a streamwise affair. If one keeps in mind this only difference, one can adopt Choice A and take the integral (26) into account later, in order to get the result for Choice B. The integrand of (26) is obviously very small at any large finite X, so that (26) is the limit of an integral of a widely spread and vanishing integrand.

For

$$-\infty \leqslant x \leqslant \xi \leqslant x_1,$$

integrating I_1 from $-\infty$ to ξ gives, with Choice A,

$$I_1 = -\iint \phi'(\xi, y, z)\,\mathrm{d}y\,\mathrm{d}z. \tag{27}$$

This integral is constant (see the development for Choice B) when $\xi \leqslant x_1$, i.e. when the region of interest does not include any part of the body. The integral can be evaluated at large values of $|\xi|$, from that part of ϕ' corresponding to a doublet in the body, and is given by (26). Thus, for Choice A,

$$I_1 = \tfrac{1}{2}(V + m_a/\rho) \qquad \text{for } \xi \leqslant x_1, \tag{28}$$

$$I_1 = \tfrac{1}{2}(V + m_a/\rho) - V(\xi) \qquad \text{for } x_1 \leqslant \xi \leqslant x_2, \tag{29}$$

$$I_1 = -\iint \phi'(\xi, y, z)\,\mathrm{d}y\,\mathrm{d}z + m_a/\rho = -\tfrac{1}{2}(V + m_a/\rho) + m_a/\rho \qquad \text{for } \xi \geqslant x_2. \tag{30}$$

For Choice B, we need to add the term (26) to the right-hand sides of the formulas above. Then I_1 is zero for $\xi \leqslant x_1$, gradually changes to $-V$ at $\xi = x_2$, and remains $-V$ thereafter, in agreement to the results given in §3. We emphasize that the term (26) is not measurable. Thus the measurable part of the drift volume is exactly the same, whether we choose procedure A or B, and the time evolution of the drift volume in the (y', z')-plane is given by (28)–(30). Equation (29) shows that for a sphere or a comparable shape of the body (i.e. excluding shapes like plates) there is a reflux upon the passage of the body through the (y', z')-plane. This is in general agreement with the finding of Eames *et al.* (1994). Note that as $\xi \to -\infty$ the I_1 given by (27) becomes less and less measurable. The same is true of the integral in (30) as $\xi \to \infty$. For an experimenter measuring with an instrument of high but not infinite precision, the measured drift volume for $\xi < x_1$ will not be constant, but will vary smoothly from zero to the value (28) in the finite part of the plane. Similarly, as $\xi \to \infty$, the measurable value of the integral in (30) will decrease from the value given for it in (30) to zero smoothly, leaving eventually only the measurable part m_a/ρ. This enables us to say that in the physical world $m_d = m_a$, so that there is a Darwin theorem.

The drift volume given by (28)–(30) gives no indication of the distribution of drift, i.e. the shape of the red surface. For that we have to calculate the drift distance and its location – that is, the values of y and z at which to 'erect' the drift distance. The distribution of the drift distance is indeed very different from that of ϕ' in (7).

The procedure for determining the shape of the drift surface is as follows.

(i) Calculate the stream functions ψ and χ, and draw the streamlines. At $x = \xi$, determine $y(\xi, \psi, \chi)$ and $z(\xi, \psi, \chi)$. At this stage, it is usually more convenient to take any y and z at $x = \xi$ and calculate ψ and χ instead.

(ii) For any streamline, i.e. for any specified value of ψ and χ, calculate the time-lag integral I_3 given by (15), with limits of integration $-\infty$ and ξ. This is the additional time Δt needed for a particle on that streamline to go from $x = -\infty$ to $x = \xi$, in comparison with a kinematic point of unit velocity to travel the same distance in x. Instead of using (15) as it is, of which each of the terms is infinite, although their difference is integrable, we may revert to

$$I_3 = -\int_{-\infty}^{\phi(\xi)} (u'/q^2)\,\mathrm{d}\phi \tag{31}$$

for each streamline. In (31) $\phi(\xi)$ is the short form for $\phi(\xi, \psi, \chi)$. We may choose to convert (31) into an integral in x, or, for axisymmetric flows, one in r.

(iii) For the same streamline, select the fluid particle at $x = \xi$, and find the location of this particle at time $t = \xi$ by calculating from its location at $x = \xi$, where $t = x = \xi + \Delta t$, to its location at $t = \xi$. Given the flow, the velocity components are known on the streamline, and what we have to do is to determine, along that streamline, the drift distance $d(\xi, \psi, \chi)$ so that

$$\Delta t = \int_{\xi-d}^{\xi} \frac{\mathrm{d}x}{u}, \tag{32}$$

where d is the drift distance. Once d is known, the location of the particle on the chosen streamline at $t = \xi$ is known, and we know where to put or 'erect' that distance to get the shape of the drift surface.

We note that the time lag can be written by virtue of (8), as

$$\int_{-\infty}^{\phi} (q'/q)^2\,\mathrm{d}\phi - \phi'(\xi, \psi, \chi) + \phi'(-\infty, \psi, \chi).$$

This form probably has more interpretive value than computational value, but it does show that unless the second term is sufficiently positive and sufficiently large, there is no local reflux. It also shows that for $\xi = \infty$ the term $\phi(\xi, \psi, \chi)$ vanishes, indicating an infinitely widely distributed ϕ' of vanishing magnitude which makes no contribution to the drift distance, even through at large ξ its integral over a (y, z)-plane at $x = \xi$ is not zero. The same thing can be said of the term $\phi'(-\infty, \psi, \chi)$. Thus, as far as the completed drift distance is concerned, only the first term above makes a contribution. This term is positive-definite, and is associated with the kinetic energy of the flow.

When the drift surface for $t = \xi_0$ (say), or for $\xi = \xi_0$ has been obtained, we may switch to the Lagrangian approach if we prefer. Then for $\xi > \xi_0$, we have, on each streamline

$$\int_{x(\xi_0)}^{x(\xi)} \frac{\mathrm{d}x}{u} = \xi - \xi_0, \tag{33}$$

where $x(\xi_0)$ is the x-position of the marked particle at time ξ_0. Once $x(\xi)$ is determined, the position of the particle on the streamline at time ξ is known. The drift surface at time ξ is thus determined.

5. Concluding remarks

The main points and results of this paper are as follows.

(i) The drift shape and the measurable part of the drift volume can both be uniquely determined. The indeterminancy of Darwin's drift integral does not matter to the physical world, in which there is a Darwin theorem, which says $m_d = m_a$.

(ii) The evolution of the drift volume as a function of time at a plane normal to the passage of the body is explicitly given. It can show a reflux as the body passes through that plane, but the measurable part eventually becomes m^a/r.

(iii) The method for determining the shape of the drift surface at any time is given.

The indeterminancy of Darwin's drift integral has been troubling us for more than four decades, and many researchers have devoted much effort in attempting to find its resolution. I do not think it is possible to show mathematically, as demanded implicitly by Benjamin (1986), that one procedure (integrating streamwise first, for instance) is the only correct and allowable way of evaluating that integral. But, as shown herein, the difference between the drift volumes obtained by two extreme procedures simply does not matter to the physical world. In the course of this work I have been understandably reminded of a saying attributed to Einstein. I do not recall the exact words, but the substance is

'Nature may be subtle, but She is not malicious.'

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